



Polynomial discrepancy of sequences¹

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Dedicated to Professor Edmund Hlawka on the occasion of his 80th birthday

Abstract

Generalizing E. Hlawka's concept of polynomial discrepancy we introduce a similar concept for sequences in the unit cube and on the sphere. We investigate the relation of this polynomial discrepancy to the usual discrepancy and obtain lower and upper bounds. In a final section some computational results are established.

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1. Introduction

We consider sequences $X = (x_n)_{n=1}^{\infty}$ of points in the unit cube $I_k = [0, 1)^k$. Let $x = X_N = \{x_1, \dots, x_N\}$ be the initial segment of the sequence X and define the discrepancy by

$$D_N(x) = \sup_{J \subset I_k} \left| \frac{1}{N} \sum_{n=1}^N \chi_J(x_n) - \lambda_k(J) \right|, \quad (1)$$

where χ_J is the characteristic function of the interval $J \subset I_k$ and λ_k denotes the k -dimensional Lebesgue measure. A sequence X is uniformly distributed if and only if

$$\lim_{N \rightarrow \infty} D_N(X_N) = 0.$$

In the theory of uniformly distributed sequences it is well known that for characterizing uniform distribution the characteristic functions of intervals can be replaced by any subset of functions which is dense in the space of all real-valued continuous functions on I_k , cf. [8, 7]. Hlawka [6] introduced the

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concept of polynomial discrepancy P_N by replacing characteristic functions by monomials. By definition

$$P_N(\mathbf{x}) = \sup_{(m_1, \dots, m_k) \in \mathbb{N}^k} \left| \frac{1}{N} \sum_{n=1}^N (x_n^{(1)})^{m_1} \cdots (x_n^{(k)})^{m_k} - \prod_{j=1}^k \frac{1}{m_j + 1} \right|. \quad (2)$$

Later Tichy [14] generalized this concept to arbitrary weighted means, where in this more general context the following bounds were established:

$$P_N(\mathbf{x}) \leq D_N(\mathbf{x}) \leq c_k \frac{1}{\log(1/P_N(\mathbf{x}))}, \quad (3)$$

c_k denoting a constant depending only on the dimension.

In the one-dimensional case Schmidt [11] showed that both inequalities are optimal in the following sense:

- (i) For any positive integer N and any set $\mathbf{x} = \{x_1, \dots, x_N\}$ of points in I_k we have $P_N(\mathbf{x}) > (1/100)(1/N)$
- (ii) For every $\varepsilon > 0$ there exists an integer N and a set $\mathbf{x} = \{x_1, \dots, x_N\}$ of points in I_k such that $P_N(\mathbf{x}) < \varepsilon$ and

$$D_N(\mathbf{x}) > e^{-1} \frac{1}{\log(1/P_N(\mathbf{x}))}.$$

In Section 2 of the present paper we extend this result to any dimension and prove the following theorem:

Theorem 1.1. *For any dimension $k \geq 1$ and any $\varepsilon > 0$, there exists an integer N and a set $\mathbf{x} = \{x_1, \dots, x_N\}$ of points in I_k such that $P_N(\mathbf{x}) < \varepsilon$ and*

$$D_N(\mathbf{x}) > c_k^* \frac{1}{\log(1/P_N(\mathbf{x}))^k}$$

with constant c_k^* depending only on the dimension.

In Section 3 we will consider uniformly distributed sequences on the d -dimensional unit sphere S^d . Several kinds of discrepancies have been introduced. The most natural concept seems to be the spherical cap discrepancy

$$D_N^C(\mathbf{x}) = \sup_C \left| \frac{1}{N} \sum_{n=1}^N \chi_C(x_n) - \sigma(C) \right|, \quad (4)$$

where a spherical cap $C = \{x: \langle x, y \rangle \geq \cos \rho\}$ is the intersection of the sphere and a half space and σ denotes the normalized surface measure; i.e. $d\sigma = (1/\omega_d) d\omega$, where ω_d denotes the surface area of S^d and ω is the standard surface measure on S^d , cf. [10]. For the case $d = 2$ Freedman [3] developed a concept based on the Green's function $G(\lambda, x, y)$ of Beltrami operator and concluded

that the quantity

$$D_N^G(\mathbf{x}) = \sup_{y \in S^d} \left| \frac{4\pi}{N} \sum_{n=1}^N G(\lambda, x_n, y) \right| \quad (5)$$

might be a suitable measure for the quality of the distribution. In fact, Hlawka [7] proved that these discrepancies are both compatible in the sense that for any sequence X

$$\lim_{N \rightarrow \infty} D_N^C(X_N) = 0 \Leftrightarrow \lim_{N \rightarrow \infty} D_N^G(X_N) = 0. \quad (6)$$

In 1991 Grabner [4] established a bound for the cap discrepancy analogous to the Erdős–Turán inequality.

For any positive integer M and constants $c_{i,d}$ depending only on the dimension d the inequality

$$D_N^C(\mathbf{x}) \leq \frac{c_{1,d}}{M+1} + \sum_{m=1}^M \left(\frac{c_{2,d}}{m} + \frac{c_{3,d}}{M+1} \right) \sum_{j=1}^{Z(d,m)} \left| \frac{1}{N} \sum_{n=1}^N S_{m,j}(x_n) \right| \quad (7)$$

holds, where $\{S_{m,j} : 1 \leq j \leq Z(d,m)\}$ denotes a set of orthonormal spherical harmonics of degree m . This observation suggests that a notion of discrepancy based on spherical harmonics might be a fruitful concept. We will investigate this idea in Section 3 and show that such a discrepancy

$$D_N^S(\mathbf{x}) = \sup_{m \geq 1} \frac{1}{m^d} \sup_{1 \leq j \leq Z(d,m)} \left| \frac{1}{N} \sum_{n=1}^N S_{m,j}(x_n) \right| \quad (8)$$

is in the above sense compatible with the cap discrepancy. More precisely, we will prove that for any set $\mathbf{x} = \{x_1, \dots, x_N\}$ of points on S^d

$$c'_d D_N^S(\mathbf{x}) \leq D_N^C(\mathbf{x}) \leq c_d D_N^S(\mathbf{x})^{1/2d}. \quad (9)$$

Application of the addition theorem for spherical harmonics easily enables us to pass over to expressions involving Legendre polynomials P_m^d ; we will consider the discrepancy

$$D_N^P(\mathbf{x}) = \sup_{m \geq 1} \frac{1}{m^d} \frac{1}{N} \sqrt{\sum_{i=1}^N \sum_{l=1}^N P_m^d(\langle x_i, x_l \rangle)}. \quad (10)$$

In the case $d=2$ a different approach was done by Cui and Freedman [2], who used elements of the theory of weighted Sobolev spaces to obtain an estimate for the approximation error in terms of the discrepancy

$$\begin{aligned} D_N^{p^*}(\mathbf{x}) &= \frac{1}{2\sqrt{\pi}N} \left[\sum_{i=1}^N \sum_{l=1}^N \sum_{m=1}^{\infty} \frac{1}{m(m+1)} P_m^d(\langle x_i, x_l \rangle) \right]^{1/2} \\ &= \frac{1}{2\sqrt{\pi}N} \left[\sum_{i=1}^N \sum_{l=1}^N \left(1 - 2 \ln \left(1 + \sqrt{\frac{1 - \langle x_i, x_l \rangle}{2}} \right) \right) \right]^{1/2}. \end{aligned}$$

Since the weighted Sobolev norm of $(1/m^2)S_{m,j}$ has a bound which is independent of m and j , it immediately follows from the main result in [2] that

$$D_N^S(\mathbf{x}) \leq c_2^* D_N^{P^*}(\mathbf{x}).$$

It is however unclear if it is possible to prove a counter-inequality.

It is not surprising that Legendre polynomials and spherical harmonics appear in expressions for discrepancy. In fact there is a relation to the construction of optimally chosen integration points, the so-called spherical designs.

A set of points $\mathbf{x} = \{x_1, \dots, x_N\}$ on S^d is called a spherical t -design if

$$\frac{1}{N} \sum_{n=1}^N p(x_n) = \int_{S^d} p(x) d\sigma(x)$$

for all polynomials (in $d+1$ variables restricted to S^d) of degree not greater than t . Now, we expect that spherical t -designs for large t have small discrepancy. In fact, we may easily deduce from the definitions that any spherical t -design \mathbf{x} verifies

$$D_N^P(\mathbf{x}) \ll \frac{1}{t^d}.$$

A similar bound is true for $D_N^{P^*}$, cf. [2]. Estimates concerning the number of points of a spherical t -design are due to Wagner, cf. [15].

Finally, we carry out some numerical experiments in Section 4.

2. Proof of Theorem 1.1

We need two lemmas. The first one gives an approximation of a step measure by a discrete measure, where a step measure is a probability measure with a step function as density function.

Lemma 2.1. *Let τ be a step measure on I_k . For any $\varepsilon > 0$ there exists a set $\{x_1, \dots, x_M\}$ of points in I_k such that for all maps $f: I_k \rightarrow \mathbb{R}$ with $\|f\|_\infty \leq 1$ and bounded variation $V(f) \leq 1$ (in the sense of Hardy and Krause), the inequality*

$$\left| \int_{I_k} f d\tau - \frac{1}{M} \sum_{m=1}^M f(x_m) \right| < \varepsilon$$

holds.

Proof. We start with a low discrepancy set of points $\mathbf{y} = \{y_1, \dots, y_N\}$ with discrepancy $D_N(\mathbf{y}) = O((\log N)^k/N)$; for instance, use the Halton sequence, cf. [8]. Now the set $\mathbf{x} = \{x_1, \dots, x_M\}$ consists of all points y_n each of them taken $[N t(y_n)]$ times, where t denotes the density function of τ . Thus,

we have

$$\begin{aligned} M &= N \sum_{n=1}^N t(y_n) + O(N) \\ &= N^2 \int_{I_k} t(x) dx + N^2 V(t) D_N(y_n) + O(N (\log N)^k) \\ &= N^2 (1 + O((\log N)^k/N)), \end{aligned}$$

where we have used the Koksma–Hlawka inequality. By a similar computation using the bounds on $|f|$ and $V(f)$ we obtain

$$\frac{1}{M} \sum_{m=1}^M f(x_m) = \frac{1}{M} \sum_{n=1}^N [N t(y_n)] f(y_n) = \int_{I_k} f d\tau + O((\log N)^k/N). \quad \square$$

For the next lemma we define the discrepancy of a step measure by

$$D(\tau) = \sup_J \left| \int_{I_k} \chi_J(x) d\tau(x) - \int_{I_k} \chi_J(x) dx \right|,$$

and analogously the polynomial discrepancy $P(\tau)$, where the characteristic functions are replaced by monomials.

Lemma 2.2. *Let L be a positive integer and $0 < \rho < 1/L$. Then there exists a step measure τ such that*

$$D(\tau) \geq \rho^k \quad \text{and} \quad P(\tau) \leq \frac{(\rho L)^L}{L}.$$

Proof. We introduce the notation

$$a_{l_1, \dots, l_k, m_1, \dots, m_k} = \int_{l_1 \rho}^{(l_1+1)\rho} \dots \int_{l_k \rho}^{(l_k+1)\rho} (x^{(1)})^{m_1} \dots (x^{(k)})^{m_k} dx,$$

where l_j and m_j are nonnegative integers and $0 \leq l_1, \dots, l_k \leq L-1$. Next we consider the following system of linear equations

$$\sum_{l_1=\dots=l_k=0}^{L-1} a_{l_1, \dots, l_k, m_1, \dots, m_k} y_{l_1, \dots, l_k} = \prod_{j=1}^k \frac{(\rho L)^{m_j+1}}{m_j+1}, \quad 0 \leq m_j \leq L-2, \quad 1 \leq j \leq k \quad (11)$$

which consists of $(L-1)^k$ equations in L^k variables y_{l_1, \dots, l_k} . Since $y_{l_1, \dots, l_k} \equiv 1$ is a solution, there exists a solution y_{l_1, \dots, l_k}^* such that

$$|y_{l_1, \dots, l_k}^* - 1| \leq 1 \quad \forall l_1, \dots, l_k \quad \text{and} \quad y_{l'_1, \dots, l'_k}^* = 1 \pm 1$$

for some l'_1, \dots, l'_k .

Now, define the density function

$$t(x) = \begin{cases} y_{l_1, \dots, l_k}^* & \text{if } l_j \rho < x^{(j)} \leq (l_j + 1)\rho, \quad 0 \leq l_j \leq L-1, \quad 1 \leq j \leq k, \\ 1 & \text{otherwise.} \end{cases}$$

We see that

$$\int_{I_k} t(x) dx = \sum_{l_1 = \dots = l_k = 0}^{L-1} y_{l_1, \dots, l_k}^* \rho^k + 1 - (\rho L)^k = 1,$$

where the last equality follows from (11) since $a_{l_1, \dots, l_k, 0, \dots, 0} = \rho^k$. The bound $D_N(\tau) \geq \rho^k$ follows immediately from the construction of the step measure. Since y_{l_1, \dots, l_k}^* is a solution of (11), the expression

$$\begin{aligned} & \int_{I_k} (x^{(1)})^{m_1} \dots (x^{(k)})^{m_k} d\tau(x) - \int_{I_k} (x^{(1)})^{m_1} \dots (x^{(k)})^{m_k} dx \\ &= \sum_{l_1 = \dots = l_k = 0}^{L-1} \int_{l_1 \rho}^{(l_1+1)\rho} \dots \int_{l_k \rho}^{(l_k+1)\rho} (x^{(1)})^{m_1} \dots (x^{(k)})^{m_k} y_{l_1, \dots, l_k}^* dx \\ & \quad - \int_0^{\rho L} \dots \int_0^{\rho L} (x^{(1)})^{m_1} \dots (x^{(k)})^{m_k} dx \\ &= \sum_{l_1 = \dots = l_k = 0}^{L-1} a_{l_1, \dots, l_k, m_1, \dots, m_k} y_{l_1, \dots, l_k}^* - \prod_{j=1}^k \frac{(\rho L)^{m_j+1}}{m_j+1} \end{aligned}$$

vanishes if all m_j are smaller than $L-1$. If this is not the case, then there exists an m_{j^*} which is at least $L-1$ and we obtain

$$\begin{aligned} & \left| \int_{I_k} (x^{(1)})^{m_1} \dots (x^{(k)})^{m_k} d\tau(x) - \int_{I_k} (x^{(1)})^{m_1} \dots (x^{(k)})^{m_k} dx \right| \\ & \leq \sum_{l_1 = \dots = l_k = 0}^{L-1} \int_{l_1 \rho}^{(l_1+1)\rho} \dots \int_{l_k \rho}^{(l_k+1)\rho} (x^{(1)})^{m_1} \dots (x^{(k)})^{m_k} |y_{l_1, \dots, l_k}^* - 1| dx \\ & \leq \int_0^{\rho L} \dots \int_0^{\rho L} (x^{(1)})^{m_1} \dots (x^{(k)})^{m_k} dx \leq \int_0^{\rho L} (x^{(j^*)})^{m_{j^*}} dx^{(j^*)} \leq \frac{(\rho L)^L}{L}, \end{aligned}$$

which gives the desired upper bound for $P(\tau)$. \square

Proof of Theorem 1.1. Combining Lemmas 2.1 and 2.2 we get for any positive integer L sets $\mathbf{x} = \{x_1, \dots, x_N\}$ of points in I_k such that $D_N(\mathbf{x}) > \rho^k - \varepsilon$ and $P_N(\mathbf{x}) < (1/L)(\rho L)^L + \varepsilon$. A suitable choice of ε leads to $D_N(\mathbf{x}) > \rho^k/2$ and $P_N(\mathbf{x}) < (2/L)(\rho L)^L$. If we set $\rho = 1/eL$ and choose L such that $1/L < \varepsilon$, we can easily deduce $P_N(\mathbf{x}) < \varepsilon$ and

$$D_N(\mathbf{x}) > c_k^* |\log P_N(\mathbf{x})|^{-k}. \quad \square$$

3. Sequences on the sphere

In this section we establish estimates for the different types of discrepancies introduced in Section 1.

Lemma 3.1. *Let $P_m^d(x)$ be the normalized d -dimensional Legendre polynomial of degree m such that $P_m^d(1) = 1$. Further let $\gamma \in (0, \pi/2]$ be a positive real number not greater than the first positive zero of $P_m^d(\cos \phi)$. Then*

$$I(\gamma) := \int_0^\gamma P_m^d(\cos \phi) (\sin \phi)^{d-1} d\phi \geq \frac{\gamma^d (\sin \gamma/\gamma)^{d-1}}{d(d+1)}.$$

Proof. For $\phi \in [0, \gamma]$ we have $\sin \phi \geq (\sin \gamma/\gamma) \phi$ and $P_m^d(\cos \phi) \geq 1 - \phi/\gamma$ since $P_m^d(\cos \phi)$ is concave in $[0, \gamma]$, cf. [1, Ch. 22]. Now an elementary computation shows

$$\begin{aligned} I(\gamma) &\geq \int_0^\gamma \left(1 - \frac{\phi}{\gamma}\right) \left(\frac{\sin \gamma}{\gamma}\right)^{d-1} \phi^{d-1} d\phi \\ &= \left(\frac{\sin \gamma}{\gamma}\right)^{d-1} \frac{\gamma^d}{d(d+1)}. \quad \square \end{aligned}$$

Corollary 3.2. *Let θ_0 be the first positive zero of $P_m^d(\cos \phi)$. Then*

$$I(\theta_0) \geq m^{-d} \frac{\pi}{2d(d+1)}.$$

Proof. Since P_m^d is up to some factor equivalent to the ultraspherical polynomial $P_m^{((d-1)/2)}$, we know from Theorem 6.21.1 in [13] and Stieltjes' bound (cf. Theorem 6.21.3 in [13]) that the first positive zero θ_0 of $P_m^d(\cos \phi)$ is not smaller than $\pi/2m$ for all $d \geq 2$ and $m \geq 1$. Therefore, we can deduce the bound by the above lemma, using $\sin(\pi t/2) \geq t$ for any $t \in [0, 1]$. \square

Now, we are able to prove the compatibility of D_N^C and D_N^S .

Theorem 3.3. *Define $D_N^C(\mathbf{x})$ and $D_N^S(\mathbf{x})$ as in (4) and (8), respectively. Then*

$$D_N^S(\mathbf{x}) \frac{\omega_{d-1}\pi}{2d(d+1)} \leq D_N^C(\mathbf{x}) \leq c_d (D_N^S(\mathbf{x}))^{1/2d}$$

holds for all sets of points $\mathbf{x} = \{x_1, \dots, x_N\}$ on S^d .

Proof. Let $C(y, \rho)$ denote the spherical cap $\{x: \|x\| = 1 \text{ and } \langle x, y \rangle \geq \cos \rho\}$, where $\|y\| = 1$ and $0 < \rho < \pi$. Applying the Funk–Hecke formula (cf. [10]), we deduce

$$\begin{aligned} \int_{S^d} \chi_{C(y, \rho)}(x) S_{m,j}(x) d\sigma(x) \\ = \int_{S^d} \chi_{[0, \rho]}(\arccos \langle x, y \rangle) S_{m,j}(x) d\sigma(x) = \beta_m S_{m,j}(y), \end{aligned} \quad (12)$$

where

$$\begin{aligned}\beta_m &= \omega_d^{-1} \omega_{d-1} \int_{-1}^1 \chi_{[0,\rho)}(\arccos t) P_m^d(t) (1-t^2)^{(d-2)/2} dt \\ &= \omega_d^{-1} \omega_{d-1} \int_0^\rho P_m^d(\cos \phi) (\sin \phi)^{d-1} d\phi.\end{aligned}$$

Specializing $\rho = \theta_0$ and using Corollary 3.2, we get

$$\beta_m \geq \omega_d^{-1} \omega_{d-1} m^{-d} \frac{\pi}{2d(d+1)}.$$

Now put $y = x_n$ in (12) and sum to obtain

$$\begin{aligned}\frac{\omega_{d-1}\pi}{\omega_d 2d(d+1)} \frac{1}{m^d} \frac{1}{N} \left| \sum_{n=1}^N S_{m,j}(x_n) \right| &\leq \beta_m \frac{1}{N} \left| \sum_{n=1}^N S_{m,j}(x_n) \right| \\ &= \left| \int_{S^d} \frac{1}{N} \sum_{n=1}^N \chi_{C(x_n,\rho)}(x) S_{m,j}(x) d\sigma(x) \right| \\ &= \left| \int_{S^d} \left[\frac{1}{N} \sum_{n=1}^N \chi_{C(x_n,\rho)}(x_n) - \sigma(C(x,\rho)) \right] S_{m,j}(x) d\sigma(x) \right|,\end{aligned}\quad (13)$$

since $\int_{S^d} S_{m,j}(x) d\sigma(x) = 0$ and $\sigma(C(x,\rho))$ does not depend on x . An application of the Cauchy–Schwarz inequality and the use of the orthonormality of the set $\{S_{m,j}\}$ immediately yields that (13) is

$$\leq D_N^C(x_n) \sqrt{\int_{S^d} |S_{m,j}(x)|^2 d\sigma(x)} = \omega_d^{-1} D_N^C(x_n).$$

This completes the proof for the lower bound.

For the upper bound we use (7). Since $Z(d,m) \leq e^d m^{d-1}$, we get for any positive real α

$$\begin{aligned}D_N^C(x_n) &\leq \frac{c_{1,d}}{[\alpha] + 1} + \sum_{m=1}^{[\alpha]} \left(\frac{c_{2,d}}{m} + \frac{c_{3,d}}{[\alpha] + 1} \right) m^d D_N^S(x_n) Z(d,m) \\ &\leq \frac{c_{1,d}}{[\alpha] + 1} + D_N^S(x_n) e^d \left[\sum_{m=1}^{[\alpha]} c_{2,d} m^{2d-2} + \frac{c_{3,d}}{[\alpha] + 1} \sum_{m=1}^{[\alpha]} m^{2d-1} \right].\end{aligned}$$

Since for any positive p

$$\sum_{m=1}^M m^p \leq M^p M = M^{p+1},$$

we obtain

$$\begin{aligned}D_N^C(x_n) &\leq \frac{c_{1,d}}{[\alpha] + 1} + D_N^S(x_n) e^d \left[c_{2,d} [\alpha]^{2d-1} + \frac{c_{3,d}}{[\alpha] + 1} [\alpha]^{2d} \right] \\ &\leq \frac{c_{1,d}}{\alpha} + c_{4,d} D_N^S(x_n) (\alpha + 1)^{2d-1}.\end{aligned}$$

To get the optimal α , we maximize the last expression, which yields

$$D_N^C(x_n) \leq c_d (D_N^S(x_n))^{1/2d},$$

where c_d depends only on the dimension. \square

Remark 1. If we square each member of inequality (13), sum over all j , and apply the addition theorem (cf. [10]), we get

$$\frac{\omega_{d-1}\pi}{\sqrt{\omega_d} 2d(d+1)} D_N^P(x_n) \leq D_N^C(x_n).$$

An upper bound in terms of the polynomial discrepancy follows in analogy to the above computations from the estimate

$$D_N^C(x_n) \leq \frac{c'_{1,d}}{M+1} + c'_{2,d} \sum_{m=1}^M m^{\frac{d-3}{2}} \frac{1}{N} \sqrt{\left| \sum_{i=1}^N \sum_{l=1}^N P_m^d(\langle x_i, x_l \rangle) \right|},$$

which was obtained by Grabner and Tichy [5].

4. Computational results

We now use the above defined types of discrepancy to compare the uniformity of the distribution of two sets of points which are believed to be very well distributed. The first was suggested by Lubotzky et al. [9] and may be generated as follows. Let A, B, C be rotations about the coordinate axes, each through an angle of $\arccos(-\frac{3}{5})$ and let R_k be the set of nontrivial reduced words in $A, B, C, A^{-1}, B^{-1}, C^{-1}$ with length $\leq k$. Further denote the elements of R_k by $\gamma_1, \dots, \gamma_N$ and for some fixed $p \in S^2$ define $x = \{\gamma_1 p, \dots, \gamma_N p\}$. Then the estimate

$$D_N^C(x) \ll \frac{(\log N)^{2/3}}{N^{1/3}}$$

holds. Based on numerical experiments the authors named above conjecture that the real order of magnitude might be $N^{-1/2}$.

The second method to generate uniformly distributed points on S^2 is to project Halton points (cf. [8]) onto the unit sphere; that is

$$x = \left(\sqrt{1 - (2t - 1)^2} \cos(2\pi\phi), \sqrt{1 - (2t - 1)^2} \sin(2\pi\phi), 2t - 1 \right),$$

where (t, ϕ) are the Halton points in $[0, 1]^2$. Similar transformations can be found in [12].

We continue with some remark on the efficient computation of D_N^P . Since the evaluation of formula (10) to compute D_N^P costs $O(N^2)$ operations, it is of little practical use to compare large sets of points. If we restrict the supremum to all m which are smaller than some small K , we obtain a lower bound for D_N^P . Computational experiments show that this bound is close to the actual value

Table 1

N	Halton points		
	D_N^P	D_N^C	$D_N^{P^*}$
7	0.19359647	0.75357343	0.07703377
37	0.05816214	0.27264030	0.02365180
187	0.00891845	0.06475563	0.00732958
937	0.00303648	0.01932728	0.00216937
4687	0.00061533	0.00432535	0.00066786
23 437	0.00016698	0.00115578	0.00020119

N	Lubotzky points		
	D_N^P	D_N^C	$D_N^{P^*}$
7	0.08571429	0.85001548	0.07238972
37	0.14162162	0.36053546	0.03956820
187	0.01074978	0.09654534	0.01163396
937	0.02834749	0.04857147	0.00796911
4687	0.00202108	0.01089805	0.00219379
23 437	0.00556241	0.00790962	0.00148915

if the points are not designed to perform especially well on polynomials. A slight modification of formula (10) leads to a reduction of the costs to $O(N)$.

We have

$$\begin{aligned}
 \sum_{i=1}^N \sum_{j=1}^N \langle x_i, x_j \rangle^k &= \sum_{i=1}^N \sum_{j=1}^N \left(\sum_{s=1}^3 x_i^{(s)} x_j^{(s)} \right)^k \\
 &= \sum_{s=0}^k \sum_{t=s}^k \frac{k!}{s!(t-s)!(k-t)!} \sum_{i=1}^N \sum_{j=1}^N (x_i^{(1)} x_j^{(1)})^s (x_i^{(2)} x_j^{(2)})^{t-s} (x_i^{(3)} x_j^{(3)})^{k-t} \\
 &= \sum_{s=0}^k \sum_{t=s}^k \frac{k!}{s!(t-s)!(k-t)!} \left(\sum_{n=1}^N (x_n^{(1)})^s (x_n^{(2)})^{t-s} (x_n^{(3)})^{k-t} \right)^2
 \end{aligned}$$

and, therefore, we can reduce the cost for the evaluation of such a lower bound for D_N^P to cK^3N , where c is some constant independent of K and N .

We use this algorithm to compute D_N^P for the two sets of points (Table 1), which were introduced above. To approximate the cap discrepancy D_N^C we take the maximal error for a large number of quasi-random caps.

We conclude that the transformed Halton sequence yields by far better distributed points than those obtained with the method of Lubotzky and all. We furthermore believe that the discrepancy D_N^P can serve as an effective qualitative measure to compare uniformly distributed sequences on S^2 .

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